A SET OF POSTULATES FOR REAL ALGEBRA, COMPRISING POSTULATES FOR A ONE-DIMENSIONAL CONTINUUM AND FOR THE THEORY OF GROUPS*

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CONTENTS.

The postulates for real algebra presented in this paper may be analysed into three groups: (1) propositions concerning the relation $<$, which, taken by themselves, form a set of independent postulates for a one-dimensional continuum, or a continuous scale; (2) propositions concerning the operation $+$, which, taken by themselves, form a set of independent postulates for the theory of groups; and (3) propositions connecting the two symbols $<$ and $+$. All these postulates taken together form a complete set of postulates for real ‡ algebra as given in §§ 1–4. Various modifications of this set are given in the later sections of the paper and in the appendix.

All these postulates are shown to be *independent*, that is, the list contains no redundancies; and the system which they determine is shown to be unique, §

* Presented to the Society at the St. Louis summer meeting, September 17, 1904, and to the Congress of Arts and Sciences, Section of Algebra and Analysis, St. Louis, September 22, 1904. Received for publication September 17, 1904.

† Received for publication October 3, 1904.

‡ A set of postulates for *complex* algebra will be presented in a subsequent paper.

§ A set of postulates having this property has been called a "categorical," as distinguished from a "disjunctive," set; see O. VEBLEN, Transactions, vol. 5 (1904), p. 346. Every proposition concerning a class K, a relation $<$, and an operation $+$, is either deducible from the postulates of this set, or in contradiction with them.

that is, there is essentially only one system (namely the system of all real numbers) in which a relation $<$ and an operation $+$ are so defined as to satisfy all the postulates (see below, theorem 30.) The existence of the system of real numbers, as built up from the positive integers by the "genetic" method of successive generalization of the number concept, * proves, moreover, the *consistency* of the postulates.

The postulates for a continuum (§ 1) are the obvious ones (see, for example, SCHÖNFLIESS's *Bericht über Mengenlehre* †), but their independence has not heretofore been established.

The postulates for the theory of groups (§ 2 and § 5) carry the analysis farther, it is thought, than the earlier sets given by the writer ‡ and by E. H. MOORE.§

The postulates for real algebra (§ 3 and § 6) are more satisfactory than the writer's earlier set ‖ in several respects; first, the separation of the postulates concerning $<$ from those concerning $+$ is now complete; secondly, the individual postulates are more nearly simple statements (and are hence more numerous); and finally, in the statement of the postulates no assumption is made in regard to the existence of any kind of numbers. This last improvement was suggested by a recent paper by BURALI-FORTI, to which further reference will be made in § 3.

On the fundamental concepts involved, namely: *class*, and *element* of a class; *dyadic relation;* and *operation*, or *rule of combination;* see Transactions, vol. 5 (1904), p. 288–290; for further bibliographical references see also Transactions, vol. 3 (1902), p. 265, and vol. 4 (1903), p. 358; and the *Theoretische Arithmetik* of STOLZ and GMEINER (1901 —).

§ 1. A SET OF POSTULATES FOR A ONE-DIMENSIONAL CONTINUUM.

In this section we consider the conditions which must be imposed upon a *class* K, and a *dyadic relation* $<$ (read: "below," or "algebraically less than"),¶ in order that K shall be a *one-dimensional continuum*, or a *continuous scale* with respect to $<$. These conditions are expressed in the following eight postulates: I, II, $R1$–$R6$ (the letter R being prefixed to indicate that the postulates concern the Relation $<$).**

* Cf. D. HILBERT, *Über den Zahlbegriff*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), part 1, pp. 180–184.

† *Ibid.*, vol. 8 (1900), part 2.

‡ E. V. HUNTINGTON, Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1901–2), pp. 296–300; Transactions, vol. 4 (1903), pp. 27–30.

§ E. H. MOORE, Transactions, vol. 3 (1902), pp. 485–492; vol. 5 (1904), p. 549.

‖ Transactions, vol. 4 (1903), pp. 358–370.

¶ Since the notion of *quantity* is not involved here, the expressions "below" and "above" (compare a thermometer scale) are clearly preferable to "less than" and "greater than." The expressions "before" and "after" may also be used to advantage.

** A summary of these postulates will be given in § 4.

The symbol $<$ here denotes a general dyadic relation, having no properties not expressly stated in the postulates. When, as in § 4, there is danger of confusion with the ordinary $<$ of arithmetic, we shall enclose the more general symbol in a circle, thus: $\otimes$; but the omission of the circle, whenever it is not absolutely necessary, will give the theorems a more familiar appearance, and thus facilitate the reading. Note that $a < b$ (a "below" b) and $b > a$ (b "above" a) denote the same relation between a and b; and that $a \leqq b$ means: $a < b$ or $a = b$.

POSTULATE I. There is an entity which belongs to the class.

This postulate tells us that K is not an "empty" class.

POSTULATE II. If a is an element of the class, there is an element b such that $a \neq b$.

This postulate excludes the trivial case of a class containing only a single element. — Postulates I and II together give us

Theorem 1. *There are at least two distinct elements in the class.*

POSTULATE $R1$. If a and b are elements of the class, and $a \neq b$, then either $a < b$ or $a > b$.

POSTULATE $R2$. If a and b are elements of the class, and $a \neq b$, then $a < b$ and $a > b$ cannot both be true.

That is, the relation $<$ is "non-symmetric" for every pair of distinct elements.

POSTULATE $R3$. If a is an element of the class, then $a < a$ cannot be true.

That is, the relation $<$ is "non-reflexive" for every element of the class. — From postulates $R1$–$R3$ we have at once:

Theorem 2. *If a and b are elements of the class, then either*

$$a = b, \qquad a < b, \qquad \textit{or} \qquad a > b;$$

and these three relations are mutually exclusive.

POSTULATE $R4$. If a, b, c are elements of the class, and $a \neq c$, then from $a < b$ and $b < c$ follows $a < c$.

In view of postulate $R2$, this postulate gives us

Theorem 3. *From $a < b$ and $b < c$ follows $a < c$, whenever a, b and c belong to the class.*

That is, the relation $<$ is "transitive."

The postulates $R1$–$R4$, with I and II, make the class a "scale" or a "simply ordered assemblage" with respect to the relation $<$; or, an assemblage possessing "*Grössencharakter*" in the sense defined by SCHÖNFLIESS (loc. cit.).

POSTULATE $R5$. If a and b are elements of the class such that $a < b$ or $a > b$, then: if $a \neq b$, there is either an element x such that $a < x$ and $x < b$, or an element y such that $a > y$ and $y > b$.

Such an element x (or y) is said to lie "between" the elements a and b; in view of theorems 1–3, we have

Theorem 4. *If a and b are two distinct elements, there is at least one, and therefore an infinite number of elements lying "between" them; and hence the class K itself is infinite.*

Thus postulate $R5$, taken with the preceding postulates, makes the class "dense" (*dicht, compact*) with respect to the relation $<$. (Without this postulate, the class might be infinite but "discrete," like the class of integral numbers, or, it might consist of merely a finite number of elements.) But a "dense" class is not necessarily continuous (as witness the system of all rational numbers with regard to $<$); we therefore add the following *postulate of continuity*, which may be stated in either of two forms, $R6$ or $R6'$:

POSTULATE $R6$. If Γ is a non-empty sub-class in K, and if there is an element c in K such that every element of Γ is $< c$, then there is an element X in K having the two following properties with regard to the sub-class Γ:

1°) if a is an element of Γ, then $a \leqq X$; while

2°) if x is any element of K which is $< X$, there is an element in Γ which is $> x$.

This element X, which is readily seen to be uniquely determined by the sub-class Γ, is called the "upper limit" of the sub-class, or sometimes its "lowest upper bound." Thus, if a subclass Γ has *any* "upper bound" c in K, it will have a "*lowest* upper bound" X. If X happens to belong to the sub-class Γ, it is its highest element; but a sub-class Γ may have an upper limit and yet not have any highest element.

In place of postulate $R6$, and as equivalent to it, we might use

Postulate $R6'$: The same as postulate $R6$ with the symbols $<$ and $>$ interchanged.

In brief, if a sub-class Γ' has *any* "lower bound" in K, it will have a "*highest* lower bound," called also its "lower limit." This element will be uniquely determined by the sub-class Γ', and if it belongs to the sub-class it will be its lowest element.

Each of these postulates, $R6$ and $R6'$ can be deduced from the other (with the aid of the preceding postulates),* and from both together we have the following theorem:

Theorem 5. *If Γ and Γ' are two non-empty sub-classes in K, such that every element of Γ is $<$ every element of Γ', then there is at least one element M, which is $\geqq$ every element of Γ and $\leqq$ every element of Γ'.*

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* Thus, suppose, in the first place, that $R6$ is given, then the proof for $R6'$ is as follows: In the hypothesis, the sub-class Γ' is a non-empty class, which has *some* lower bound, say c'; we are to prove that it has *a highest* lower bound, X'. To show this, we define a sub-class Γ as the class of all those elements in K which are below every element in Γ'. This class Γ will be a non-empty class, since it contains the element c', and it will have some upper bound, since any element of the non-empty class Γ' will answer that purpose. Therefore, by postulate $R6$, the sub-class Γ will have an upper limit, X. It is then easy to show that this element X will be the required lower limit of the sub-class Γ'.—In like manner, $R6$ could be deduced from $R6'$.

For, Γ will have its upper limit, X, and Γ' will have its lower limit, X', by postulates $R6$ and $R6'$, and $X \leqq X'$. Then take $M = X$, or X', or any element which may lie between X and X'.

In particular, if the two sub-classes Γ and Γ' make up, together, the whole class K, the upper limit of Γ will coincide with the lower limit of Γ', there will be only one element M, and this will be either the highest element in Γ or the lowest element in Γ'.*

Thus every system $(K, <)$ which satisfies postulates I, II, $R1$–$R6$ will have the property of continuity as defined by DEDEKIND in his *Stetigkeit und irrationale Zahlen*, and may be called a *one-dimensional continuum* or a *continuous scale*, with respect to the relation $<$. [See below, end of § 1.]

Moreover, these eight postulates I, II, $R1$–$R6$, as will be shown in § 4, are *independent* of each other; that is, no one of them can be deduced from the remaining seven.

In conclusion, we notice that a one-dimensional continuum, or continuous scale, as defined by postulates I, II, $R1$–$R6$, may be either limited or unlimited, in either direction. To complete the discussion of the present section, therefore, we add the two following postulates, although they will not be required in the remainder of this paper:

Postulate $R7$. If a is an element of the class, and if there is any element $b \neq a$, then there is an element x such that $x < a$;

Postulate $R8$. If a is an element of the class, and if there is any element $b \neq a$, then there is an element y such that $y > a$.

Hence, the class has neither a lowest nor a highest element, but is unlimited in both directions.

The ten postulates: I, II, $R1$–$R8$, form a set of independent postulates (see § 4), which define completely the properties of an *unlimited* continuous scale with respect to $<$. For, if C and C' are two systems $(K, <)$ which satisfy all these postulates, they clearly can be brought into one-to-one correspondence † in such a manner that if $a < b$ in C, then $a' < b'$ in C', where a' and b' are the elements which correspond to a and b respectively. Hence the set is "categorical," ‡ that is, every proposition concerning K and $<$ must either be deducible from the postulates of the set, or be in contradiction with them.

[It should be noticed that these postulates, in spite of the fact that they form a categorical set, are hardly sufficient to define a continuum in the ordinary *geometric* sense of the word. For example, the class of positive real numbers

* Notice also that postulate $R6$ can be deduced from theorem 5 (with the aid of the preceding postulates), by taking as Γ' the subclass of all the elements of K which are $>$ every element of Γ.

† And that in an infinite number of ways.

‡ O. VEBLEN: loc. cit.

between 0 and 3, omitting the numbers x for which $1 < x \leqq 2$, is a class which satisfies all the conditions but is not a geometric continuum in any ordinary sense. All we can say is that it is *equivalent* to a continuum *as far as our relation $<$ is concerned.**]

§ 2. A SET OF POSTULATES FOR ABELIAN GROUPS.†

In this section we consider the condition which must be imposed upon a *class* K, and an *operation* or *rule of combination* $+$ (read: "plus"), in order that K shall be an *abelian group* with respect to $+$. These conditions are expressed in the following eight postulates: I, II, $A1$–$A6$ (the letter A being prefixed to indicate that the postulates concern Abelian groups, or if one will, the operation of Addition).‡

The symbol $+$ here denotes a general operation, having no properties not expressly stated in the postulates. When, as in § 4, there is danger of confusion with the ordinary $+$ of arithmetic, we shall enclose the more general symbol in a circle, thus: $\oplus$.

POSTULATE I. There is an entity which belongs to the class.

POSTULATE II. If a is an element of the class, there is an element b such that $a \neq b$.

These postulates serve to exclude the cases of an "empty" class, and a class containing a single element. Together they give

Theorem 1. *There are at least two distinct elements in the class.*

POSTULATE $A1$. If a and b are elements of the class, then $a + b$ is an element of the class.

This postulate states the fundamental property of a *group*, the element $a + b$ being called, for our present purpose,§ the "sum" of the elements a and b.

POSTULATE $A2$. If a, b, $a + b$, and $b + a$ are elements of the class, then

$$a + b = b + a.$$

This is the "commutative law" for the operation $+$, and is the characteristic property of abelian groups as distinguished from groups in general.

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* On G. CANTOR's definition of a continuum, which differs somewhat from DEDEKIND's definition, see B. RUSSELL, *Principles of Mathematics*, vol. 1 (1903), chap. 36.

† For a further analysis of the postulates for the theory of groups, see § 5.

‡ A summary of these postulates will be given in § 4.

§ If we interpreted the operation $\oplus$ as "multiplication," as one usually does in the general theory of abstract groups, we should call $a \oplus b$ the "product" of a and b. So below, instead of the "zero-element" (0), the "negative of a" ($-a$), the operation of "subtraction" ($b - a$), etc., we should speak of the "unit element" (1), the "reciprocal of a" (a^{-1}), the operation of "division" (b/a), etc. Again, if we were dealing with groups of transformations, we should call 1 the "identical transformation," or the "identity," and a^{-1} the "inverse of a." Compare § 5.

POSTULATE $A3$. If a, b, c, $a+b$, $b+c$, $(a+b)+c$, and $a+(b+c)$ are elements of the class, then

$$(a+b)+c=a+(b+c).$$

This is the "associative law" for the operation $+$. These three postulates give us

Theorem 6.* (*a*) *The operation* $+$ *is always possible within the class, and it obeys* (*b*) *the commutative law and* (*c*) *the associative law.*

POSTULATE $A4$. If $a, x, y, a+x$, and $a+y$ are elements of the class, then from $a+x=a+y$ follows $x=y$.

Hence, in view of this commutative law, we have

Theorem 7. *A change in either* a *or* b *alone produces a change in* $a+b$.

POSTULATE $A5$. If there is any element in the class, then there is an element 0 such that $0+0=0$.

It is easy to show that for any such element 0, and for every element a, $a+0=0+a=a$; and hence that the element 0 is uniquely determined.† Thus we have the theorem:

Theorem 8. (*a*) *There is a uniquely determined element* 0 *such that* $0+0=0$; (*b*) *for every element* a

$$a+0=a \qquad \textit{and} \qquad 0+a=a;$$

and (*c*) *if* $a+x=a$ *or* $x+a=a$, *then* $x=0$.

This element 0 may be called, for our present purpose, the "zero-element" of the class, the properties in theorem 8 being the "additive" properties of zero. When, as in § 4, there is any danger of confusing this symbol with the ordinary 0 of arithmetic, we shall enclose it in a circle, or (which is typographically more convenient) denote the element in question by z (the initial letter of "zero"), or by i (the initial letter of "identity").‡

POSTULATE $A6$. If there is a uniquely determined element 0 such that $0+0=0$, then for every element a there is an element $\bar{a}$ such that $a+\bar{a}=0$.

In view of postulate $A4$, this element $\bar{a}$ is uniquely determined by a, and may be called the "negative of a." ‡

Further, if we take $x=\bar{a}+b$, we have

$$a+x=a+(\bar{a}+b)=(a+\bar{a})+b=0+b=b;$$

hence,

* The theorem-numbers 2–5 have been used in § 1.

† Thus, if $z+z=z$, we have $z+(a+z)=z+(z+a)=(z+z)+a=z+(a)$, whence, by postulate $A4$, $a+z=a$, and, by the commutative law, $z+a=a$. Hence, if z_1 and z_2 were two elements such that $z_1+z_1=z_1$ and $z_2+z_2=z_2$, then $a+z_1=a$ and $b+z_2=b$, for all values of a and b; whence, taking $a=z_2$ and $b=z_1$, we should have $z_2+z_1=z_2$ and $z_1+z_2=z_1$; or, $z_1=z_2$.

‡ Compare footnote under postulate $A1$.

Theorem 9. (*a*) *Every two elements, a and b, determine uniquely a third element x, denoted by b — a, such that*

$$a + (b - a) = b\,;$$

(*b*) *in particular,*

$$a - a = 0\,;$$

and (*c*) *the element* 0 — *a is usually abbreviated into* — *a, so that*

$$a + (-a) = 0\,.$$

The element $b - a$ is called the "difference" b minus a, and the operation of finding it, the "subtraction" of a from b.*

From theorems 1, and 6–9, we see that *every system* (K, +) *which satisfies the eight postulates*, I, II, *A*1–*A*6, *is an abelian group with respect to* +, according to the usual definition.† These eight postulates, as will be shown in § 4, are *independent* of each other; that is, no one of them can be deduced from the other seven. The complete theory of abelian groups would contain all the propositions which follow from these eight postulates by logical deduction.

In conclusion, it should be noticed that the eight postulates of § 2 form a "disjunctive," not a "categorical," set; ‡ for an abelian group may contain any finite number of elements, or be infinite; and even if the number of elements in two groups is the same, the groups are not necessarily isomorphic; hence there are many propositions concerning K and + which are neither deducible from these postulates, nor in contradiction with them.

§ 3. A SET OF POSTULATES FOR REAL ALGEBRA.

In this section, which forms the main part of the present paper, we consider the conditions which must be imposed upon a *class K*, with a *dyadic relation* < (read: "below," or "algebraically less than"), and an *operation* + (read: "plus"), in order that K shall be the class of *all real variables* with respect to < and +. (On the use of the symbols < and +, or ⊘ and ⊕, see the opening paragraphs in § 1 and § 2.) These conditions are expressed in sixteen postulates, § which fall into four groups as follows:

POSTULATES I–II. (As in § 1 or § 2.)

These postulates give the class at least two distinct elements.

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* Compare footnote under postulate *A*1.

† H. WEBER, *Algebra*, vol. 2. Cf. E. V. HUNTINGTON, and E. H. MOORE, loc. cit.; and also G. A. MILLER, *Report on the groups of an infinite order*, Bulletin of the American Mathematical Society, vol. 7 (1900–1), pp. 121–130.

‡ O. VEBLEN, loc. cit.

§ A summary of these postulates is given in § 4.

POSTULATES $R1$–$R6$. (As in § 1.)

These postulates, with I and II, make the class a *continuous scale*, or a *one-dimensional continuum*, with respect to the relation $<$.

POSTULATES $A1$–$A6$. (As in § 2.)

These postulates, with I and II, make the class an *abelian group* with respect to the operation $+$.

POSTULATE $RA1$. If there is a uniquely determined element 0 such that $0 + 0 = 0$; and if a, b, and $a + b$ are elements of the class; and if $a > 0$ and $b > 0$; then $a + b > a$.

POSTULATE $RA2$. If there is a uniquely determined element 0 such that $0 + 0 = 0$; and if a, b, and $a + b$ are elements of the class; and if $a < 0$ and $b < 0$; then $a + b < a$.

These last two postulates serve to connect the symbols $<$ and $+$. In view of postulates I, II, and $A1$–$A5$, they give us at once:

Theorem 10. *If a and b are both above* 0, *then $a + b$ is above either of them; and if a and b are both below* 0, *then $a + b$ is below either of them.*

These sixteen postulates: I–II, $R1 - R6$, $A1 - A6$, $RA1$ and $RA2$, are *independent*, as will be shown in § 4, so that no one of them can be deduced from the remaining fifteen. On the other hand, these sixteen postulates, or the theorems 1–10 which follow from them,* are *sufficient to define completely the algebra of a real variable.* The remainder of this section will be devoted to the proof of this latter statement, the necessary preliminary theorems, besides the theorems 1–10, grouping themselves under the following heads: 1) positive and negative elements; 2) multiples, with remarks on BURALI-FORTI's definition of number; 3) ARCHIMEDES' principle; and 4) submultiples, with the theorem of infinite divisibility.

Positive and negative elements.

The following theorems 11–17 are proved without the use of postulate $R5$ (on density) or postulate $R6$ (on continuity).

Theorem 11. *If a and b are both* > 0, *then* $a + b > 0$; *and if a and b are both* < 0, *then* $a + b < 0$. (By 10 and 3.)

Theorem 12. *If* $a > 0$, *then* $-a < 0$; *and if* $a < 0$, *then* $-a > 0$. (By 11 and 9.)

Thus the class K is composed of three mutually exclusive, non-empty sub-classes: 1) the element 0; 2) the elements above 0, called the *positive* elements; 3) the elements below 0, called the *negative* elements. (By I, II, 12 and 3.)

Theorem 13. *If* $x > 0$, *then* $a + x > a$; *if* $x < 0$, *then* $a + x < a$.

The first part is clearly true when $a = 0$, and (by 10) when $a > 0$; the proof for the remaining case, $a < 0$, is as follows: Suppose $a + x \leqq a$, while

* It should be noted that the sixteen postulates are clearly deducible from the theorems 1–10.

$x > 0$ and $a < 0$; then $a + x < 0$, by 3, and $-x < 0$, by 12; hence by 10, $(a + x) + (-x) < a + x$, or, by 6 and 9, $a < a + x$, which contradicts the supposition. The second part may be proved in a similar way.

Theorem 14. *If $a < b$ there is a positive element x such that $a + x = b$ and a negative element y such that $a = b + y$; and conversely, if $x > 0$ in $a + x = b$, or if $y < 0$ in $a = b + y$, then $a < b$.* (By 9 and 13.)

Theorem 15. *If $x < y$, then $a + x < a + y$, and conversely.*

For, by 14, take $w < 0$ so that $x = y + w$. Then, by 13,

$$(a + y) + w < (a + y),$$

whence $a + (y + w) < a + y$, or $a + x < a + y$. The converse is proved indirectly, as usual.

Theorem 16. *If $a < b$ and $x < y$, then $a + x < b + y$.*

For, $a + x < a + y$, and $a + y < b + y$, by 15 and the commutative law; hence the theorem, by 3.

Theorem 17. *There is no highest element, and no lowest element, in the class.* (By 13.)

If we admit also the postulate of density ($R5$), we have further:

Theorem 18. *There is no highest or lowest element in the sub-class of positive elements, or in the sub-class of negative elements.*

Multiples.

The theory of the multiples of an element a is closely connected with the theory of the (finite) ordinal numbers (or positive integers). In fact C. Burali-Forti, in a recent memoir,* has shown that the theory of multiples can be developed without presupposing any knowledge of these numbers, and that the class of numbers can then be defined by means of the theory of multiples. For the present purpose, however, it seems best to assume the ordinal numbers as known, especially in view of a valuable discussion of Burali-Forti's work by L. Couturat, in a memoir † which came into my hands while writing this paper.

In order to state precisely what is involved in thus assuming the ordinal numbers, it may be well to recall the following facts.

The class of ordinal numbers, however it may be defined, is a class N which possesses the following characteristic properties, due to Peano: ‡

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* C. Burali-Forti, *Sulla teoria generale delle grandezze e dei numeri*, Atti della R. Accademia delle Scienze di Torino, vol. 39 (1903–4), pp. 256–272. On Burali-Forti's proof of the commutative law, see footnote below, in § 6.

† L. Couturat, *Les principes des mathématiques; V: L' idée de grandeur*, Revue de Métaphysique et de Morale, vol. 12 (1904), pp. 675–698.

‡ For a brief account of these postulates, with bibliographical references, see Bulletin of the American Mathematical Society, ser. 2, vol. 9 (1902–3), pp. 41–46. An extended discussion is found in B. Russell's *Principles of Mathematics*, vol. 1 (1903), chap. 14. A revised list, in which the number of postulates is reduced to four, has been given by A. Padoa, *Théorie des nombres entiers absolus*, Revue de Mathématiques, vol. 8 (1902), p. 48.

1. The class N contains a peculiar element, called the "first" ordinal number, and denoted by 1.

2. Every element m of the class N determines uniquely another element of N called the number "next following" the number m, and denoted by m^+, or seq m. ($1^+ = 2$; $2^+ = 3$; etc.)

3. For any number m, m^+ is different from 1.

4. If $m^+ = n^+$, then $m = n$.

5. If S is any class such that: (a) the number 1 belongs to S, and (b) the number m^+ belongs to S whenever the number m belongs to S; then every element of N belongs to S.

This list of five postulates comprises as much of the theory of the ordinal numbers as we need to assume in the present paper.

On the basis of these fundamental propositions, the *sum*, $p + q$, of two numbers, p and q, is the number defined by the following recurrent formulæ:

$$p + 1 = p^+,\ p + 2 = (p + 1) + 1,\ p + 3 = (p + 2) + 1,\ \cdots,$$
$$p + (k + 1) = (p + k) + 1;$$

whence, by mathematical induction, $p + q = q + p$.

Their *product*, pq, is defined by the following recurrent formulae:

$$1p = p,\ 2p = 1p + p,\ 3p = 2p + p,\ \cdots, (k + 1)p = kp + p;$$

whence, by mathematical induction, $pq = qp$.

Finally, if $p \neq q$, then there is either a number $\lambda (= p - q)$ such that $p = q + \lambda$, or else a number $\mu (= q - p)$ such that $q = p + \mu$, and not both; in the first case we write $p > q$ (p "greater than" q), in the second case, $p < q$ (p "less than" q).

Making use of these properties of the ordinal numbers, we may now define the multiples of any element of our abstract class K. But we must be careful to notice that the symbols $+$ and $<$, as used between two ordinal numbers, have nothing to do with the general symbols $+$ and $<$ (or $\oplus$ and $\otimes$) as used between the elements of K.

We define the mth *multiple*, ma, *of any element* a, m being any ordinal number, by the following recurrent formulæ:

$$1a = a,\ 2a = 1a + a,\ 3a = 2a + a,\ \cdots,\ (k + 1)a = ka + a.$$

Hence, if p and q denote ordinal numbers, and a an element of K, we have at once:

Theorem 19. 1°) $pa + qa = (p + q)a$;

2°) $q(pa) = p(qa) = (qp)a = (pq)a = pqa$.

This theorem depends only on postulates I, II, and $A1$; the following theorems

20–22, however, require all the postulates except $R5$ (on density) and $R6$ (on continuity):

Theorem 20. *If a is above* 0, *the successive multiples of a form an ascending series:* $(m+1)a > ma$; *if a is below* 0, *the successive multiples of a form a descending series:* $(m+1)a < ma$; *and if* $a = 0$, *every multiple of a is* 0 : $ma = 0$.

In particular, if $a \neq 0$, $pa = qa$ *when and only when* $p = q$.

Theorem 21. *If* $a < b$, *then* $ma < mb$, *and conversely.*

For, if $a < b$, then $a = b + x$, where $x < 0$, by 14; hence $ma = mb + mx$, where $mx < 0$, by 20. Therefore $ma < mb$, by 14. Conversely, if $ma < mb$, then $a < b$, since $a \geqq b$ leads to contradictions.

Theorem 22. *If* $ma = mb$, *then* $a = b$. (From 21.)

The principle of Archimedes.

The following theorem requires all the postulates except the postulate of density ($R5$).

Theorem 23. *If a and b are both above* 0, *there are multiples of a which are above b; and if a and b are both below* 0, *there are multiples of a which are below b.* (Principle of ARCHIMEDES.)

The proof of the first part is as follows: Suppose every multiple of a were $\leqq b$; then, by the postulate of continuity ($R6$), the class of multiples of a would have an upper limit, X, having the following properties:

1°) $na \leqq X$, for every ordinal number n (in particular, $a < X$); and

2°) if $x < X$, there is some multiple of a, say $n'a$, which is $> x$.

By theorem 14, take $x < X$ so that $x + a = X$; then by 2°), $n'a > x$ for at least one value of n'. Hence, by theorem 15 and the commutative law, $n'a + a > x + a$, whence $(n' + 1)\,a > X$, which contradicts 1°).

The second part is proved in a similar way.

Submultiples.

The following lemma requires all the postulates except $R6$ (on continuity); while the theorem 24 to which it leads, requires the whole list.

Lemma. If m is any ordinal number, then, if a is positive, there is a positive element x such that $mx < a$; and if a is negative, there is a negative element y such that $my > a$.

Proof. The lemma is clearly true when $m = 1$, by 18, and if it is true when $m = n$, it will be true when $m = n + 1$. For (considering the case $a > 0$), if $nx < a$, then there is a positive ξ such that $nx + \xi = a$, by 14, and a positive x' such that $x' < x$ and $x' < \xi$, by 18. Then $nx' < nx$ by 21, whence, $nx' + x' < nx + \xi$, by 16, or $(n+1)x' < a$. Therefore the lemma is true for every value of m.—Similarly for the case $a < 0$.

Theorem 24. *If m is any ordinal number, then every element a determines uniquely an element denoted by a/m, called the mth submultiple of a, such that*

$$m(a/m) = a.$$

In particular:

$$a/1 = a \qquad \text{and} \qquad 0/m = 0.$$

Proof. We notice in the first place, by 22, that this element a/m, if it exists at all, will be uniquely determined by a and m. To prove the existence of any such element (for a positive a), we define two subclasses: Γ, containing all the positive elements y for which $my < a$; and Γ', containing all the positive elements y' for which $a < my'$. Now Γ and Γ' are non-empty classes, by the lemma just proved, and 18; and every element in Γ is $<$ every element in Γ', by 21. Hence, by theorem 5, there is a positive element X such that: 1°) every element in $\Gamma \leqq X$, and 2°) $X \leqq$ every element in Γ'. This element X is the required element a/m. For if mX were $< a$, there would be an element y in Γ which is $> X$, in contradiction with 1°)*; and if mX were $> a$, there would be an element y' in Γ' which is $< X$, in contradiction with 2°).† Therefore $mx = a$. — The proof is similar for a negative a.

Theorem 25. *If p and q are ordinal numbers such that $p < q$, then if a is above* 0, *$a/p > a/q$, and if a is below* 0, *$a/p < a/q$.*

In other words, if $a \neq 0$, increasing the value of m brings the element a/m nearer to 0. In particular, if $a > 0$, $a > a/m > 0$, and if $a < 0$, $a < a/m < 0$; where $m > 1$. (Proof indirect, by 20, 21 and 24.)

Theorem 26. *If a and b are both above* 0, *there are submultiples of a which are below b; and if a and b are both below* 0, *there are submultiples of a which are above b.* (Proof indirect, by 23 and 24.)

Theorem 27. *If p and q denote ordinal numbers, and a an element of K, then $p(a/q) = (pa)/q$.* (By 24.)

Hence we may denote either member of this equation by pa/q.

Any element of the form pa/q, where p and q are ordinal numbers, is called a *rational fraction* of a. In particular, $p0/q = 0$.

Theorem 28. *If m, p, q, p', q', are ordinal numbers, then:*

(*a*) $mpa/mq = pa/q$.

(*b*) $pa/q + p'a/q' = (q'p + qp')a/qq'$.

(*c*) $pa/q = p'a/q'$ when and only when $q'p = qp'$.

Theorem 29. *If a is above* 0 [*below* 0], *then $pa/q < p'a/q'$ when and only when $q'p < qp'$* [*or $q'p > qp'$*]. (Proof by 20, 21 and 24.)

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* Take $\xi > 0$ so that $mX + \xi = a$, by 14, and $\eta > 0$ so that $m\eta < \xi$, by the lemma, and let $y = X + \eta$. Then $my = mX + m\eta < mX + \xi = a$.

† Take $\xi' < 0$ so that $mX + \xi' = a$, by 14, and $\eta' < 0$ so that $m\eta' > \xi'$, by the lemma, and let $y' = X + \eta'$. Then $my = mX + m\eta' > mX + \xi' = a$.

The theorems 1–29 put us now in a position to establish the fact that the postulates of § 3 are sufficient to define completely the algebra of a real variable; this sufficiency is formulated in the following theorem:

Theorem 30. *Any two systems*, $(K, <, +)$ *and* $(K', <, +)$, *which satisfy the sixteen postulates of* § 3 *are* EQUIVALENT *with respect to* $<$ *and* $+$; *that is, they can be brought into one-to-one correspondence* in such a way that*

1°) *from* $a < b$ *follows* $a' < b'$, *and conversely; and*

2°) *from* $a + b = c$ *follows* $a' + b' = c'$, *and conversely;*

where a', b', c' *are the elements in* K' *which correspond to the elements* a, b, c *in* K.

Proof. First make the zero-element in K correspond to the zero-element in K'.

Next, to bring the positive elements of K into one-to-one correspondence with the positive elements of K', choose any positive element E in K, and any positive element E' in K', and call E and E' corresponding elements; then if any positive element a is given, the corresponding positive element a' is determined as follows: consider in K the subclass Γ composed of all rational fractions of E which are $< a$; form in K' the subclass Γ' composed of the *same* rational fractions of E'; then the upper limit of Γ' is the required element a'. Similarly if a' is given and a required.

Finally, bring the negative element of K into one-to-one correspondence with the negative elements of K' by making a correspond to a' whenever $-a$ correspond to $-a'$.

The correspondence as thus established is readily shown to have the properties 1°) and 2°), by the aid of the theorems in regard to rational fractions.

Note on multiplication.

The deductions from the postulates have been carried, in this section, only so far as was necessary for the proof of theorem 30. The further development of the algebra depends on the definition of the *product* of two elements, with respect to an arbitrarily chosen (positive) "unit element."† [The arbitrary choice of this "unit element" is of great significance in the *theory of measurement*, into which I shall not enter here.]

Multiplication may also be regarded as a fundamental operation whose properties are determined by postulates, as is done below, in the appendix.

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* And that in an infinite number of ways, since the "unit elements" E and E' in the proof may be chosen in an infinite number of ways.

† Cf. E. V. HUNTINGTON, Transactions of the American Mathematical Society, vol. 4 (1903), p. 365.

§ 4. Summary of the postulates, and proof of their independence.

In the following summary of the postulates of § 3 for real algebra, each postulate is stated twice, first in the precise and convenient symbolism * of Peano's *Logica Mathematica*, and secondly in common language. To avoid confusion in the proofs of independence, we enclose the general symbols $<$ and $+$ in circles, and write z in place of 0.

Postulate I. $\exists K$;

that is, there is an entity which belongs to the class.

Postulate II. $a \epsilon K . \supset . \exists K \frown b \ni [a \neq b]$;

that is, if a is an element of the class, there is an element b such that $a \neq b$.

Postulate *R*1. $a, b \epsilon K . a \neq b . \supset : a \textcircled{<} b . \smile . a \textcircled{>} b$;

that is, if a and b are elements of the class, and $a \neq b$, then either $a \textcircled{<} b$ or $a \textcircled{>} b$.

Postulate *R*2. $a, b \epsilon K . a \neq b . \supset : a \textcircled{<} b . a \textcircled{>} b . = . \Lambda$;

that is, if a and b are elements of the class, and $a \neq b$, then $a \textcircled{<} b$ and $a \textcircled{>} b$ cannot both be true.

Postulate *R*3. $a \epsilon K . \supset . a \textcircled{<} a . = . \Lambda$;

that is, if a is an element of the class, then $a \textcircled{<} a$ cannot be true.

Postulate *R*4. $a, b, c \epsilon K . a \neq c . a \textcircled{<} b . b \textcircled{<} c . \supset . a \textcircled{<} c$;

that is, if a, b, c are elements of the class, and $a \neq c$, then from $a \textcircled{<} b$ and $b \textcircled{<} c$ follows $a \textcircled{<} c$.

Postulate *R*5. $a, b \epsilon K : a \textcircled{<} b . \smile . a \textcircled{>} b : a \neq b : \supset :$

$$\exists K \frown x \ni [a \textcircled{<} x . x \textcircled{<} b] . \smile . \exists K \frown y \ni [a \textcircled{>} y . y \textcircled{>} b];$$

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*See Burali-Forti. *Logica Matematica* (*Manuali Hoepli*, 1894), or the introductions to the several volumes of the *Formulaire de Mathématiques* (1895–1903). In reading statements written in these symbols, look first for the *dots*, which occur in groups of one, two, or more, and divide the sentence into its component parts; the larger the number of dots, the more important the point of division. Next find the symbol $\supset$, which is the sign of illation ; thus, "$a . b . \supset . c$" means "*if* the propositions a and b are true, *then* the proposition c is true ;" or, "the truth of propositions a and b *implies* the truth of propositions c." For the other symbols used in this paper, the following glossary will be sufficient :

"$a . \smile . b$" means "either a or b."

"$a \epsilon K$" means "a belongs to the class K ;" "$a, b \epsilon K$" means "a and b both belong to the class K. (The "ϵ" is the initial letter of "ἐστί.")

"$\exists K$" means "there exists an element of the class K." "$\ni$" means "such that."

"$\exists K \frown a \ni$" means "there exists an element a, in the class K, such that."

"$\Gamma \epsilon$ Cls' K" means "Γ is a sub-class in the class K."

"$a . = . \Lambda$" means "the proposition a is false."

"Hp" means "hypothesis."

that is, if a and b are elements of the class such that $a ⧀ b$ or $a ⧁ b$, then: if $a \neq b$, there is either an element x such that $a ⧀ x$ and $x ⧀ b$, or an element y such that $a ⧁ y$ and $y ⧁ b$.

Postulate $R6$. $\Gamma \epsilon \operatorname{Cls}' K . \exists \Gamma . \exists K \frown c \ni [\alpha \epsilon \Gamma . \supset . \alpha ⧀ c] . \supset .$

$$\exists K \frown X \ni [\alpha \epsilon \Gamma . \supset . \alpha \overset{⧀}{=} X : x \epsilon K . x ⧀ X . \supset . \exists \Gamma \frown \xi \ni (\xi ⧁ x)];$$

that is, if Γ is a non-empty sub-class in K, and if there is an element c in K such that every element of Γ is $⧀ c$, then there is an element X in K having the two following properties with regard to the sub-class Γ:

1°) if α is an element of Γ, then $\alpha \overset{⧀}{=} X$; while

2°) if x is any element of K which is $⧀ X$, there is an element in Γ which is $⧁ x$.

Postulate $A1$. $a, b \epsilon K . \supset . a \oplus b \epsilon K;$

that is, if a and b are elements of the class, then $a \oplus b$ is an element of the class.

Postulate $A2$. $a, b, a \oplus b, b \oplus a \epsilon K . \supset . a \oplus b = b \oplus a;$

that is, if $a, b, a \oplus b$, and $b \oplus a$ are elements of the class, then $a \oplus b = b \oplus a$.

Postulate $A3$. $a, b, c, a \oplus b, b \oplus c, (a \oplus b) \oplus c, a \oplus (b \oplus c) \epsilon K . \supset .$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c);$$

that is, if $a, b, c, a \oplus b, b \oplus c, (a \oplus b) \oplus c$, and $a \oplus (b \oplus c)$ are elements of the class, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

Postulate $A4$. $a, x, y, a \oplus x, a \oplus y \epsilon K . a \oplus x = a \oplus y . \supset . x = y;$

that is, if $a, x, y, a \oplus x$, and $a \oplus y$ are elements of the class, then from $a \oplus x = a \oplus y$ follows $x = y$.

Postulate $A5$. $\exists K . \supset . \exists K \frown z \ni (z \oplus z = z);$

that is, if there is any element in the class, then there is an element z such that $z \oplus z = z$.

Postulate $A6$. $\exists K \frown z \ni (z \oplus z = z) : z_1 \oplus z_1 = z_1 \cdot z_2 \oplus z_2 = z_2 . \supset . z_1 = z_2 : a \epsilon K : \supset :$

$$\exists K \frown \bar{a} \ni (a \oplus \bar{a} = z);$$

that is, if there is a uniquely determined element z such that $z \oplus z = z$, then for every element a there is an element $\bar{a}$ such that $a \oplus \bar{a} = z$.

Postulate $RA1$. Hp A6. $a, b, a \oplus b \epsilon K . a ⧁ z . b ⧁ z . \supset . a \oplus b ⧁ a;$

that is, if there is a uniquely determined element z such that $z \oplus z = z$; and if a, b, and $a \oplus b$ are elements of the class; and if $a ⧁ z$ and $b ⧁ z$; then $a \oplus b ⧁ a$.

POSTULATE $RA2$. Hp A6. $a, b, a \oplus b \epsilon K. a \otimes z. b \otimes z. \supset. a \oplus b \otimes a$;

that is, if there is a uniquely determined element z such that $z \oplus z = z$; and if a, b, and $a \oplus b$ are elements of the class; and if $a \otimes z$ and $b \otimes z$; then $a \oplus b \otimes a$.

This completes the list of postulates in § 3.

To show the mutual independence of these sixteen postulates—that is, to show that the list contains no redundancies—we exhibit, for each postulate, an interpretation of K, $\otimes$, and $\oplus$ which satisfies all the other postulates, but not the one in question. Such postulate, then, cannot be a consequence of the other fifteen; for, if it were, every system $(K, \otimes, \oplus)$ which had the other fifteen properties would have this property also, which is not the case.*

In constructing these systems, we assume the existence of the real numbers, as defined in the usual way by successive generalizations of the number-concept; and the symbols $<$, $+$, and 0 are used, in this section, only in their ordinary arithmetic meaning. The proof-systems are the following:

For I. $K =$ an "empty" class.

For II. $K =$ a class of a single element a. $a \otimes a$ false. $a \oplus a = a$.

For $R1$. $K =$ all real numbers. $(a \otimes b) = (a < b)$ when a and b are both positive or both negative; otherwise $a \otimes b$ is false. $\oplus = +$.

For $R2$. $K =$ all real numbers. $(a \otimes b) = (a \neq b)$. $\oplus = +$.

For $R3$. $K =$ all real numbers. $(a \otimes b) = (a \leqq b)$. $\oplus = +$.

For $R4$. $K =$ a class of three elements, $0, 1$, and 2. $0 \otimes 1, 1 \otimes 2, 2 \otimes 0$. $a \oplus b = a + b$ when $a + b \leqq 2$, and $= a + b - 3$ when $a + b > 2$; in other words, $a \oplus b \equiv a + b \pmod 3$.

For $R5$. $K =$ all integral numbers. $\otimes = <$. $\oplus = +$.

For $R6$. $K =$ all rational numbers. $\otimes = <$. $\oplus = +$.

For $A1$. $K =$ all real numbers. $\otimes = <$. $a \oplus b = a + b$ when $a + b = 0$, otherwise $a \oplus b =$ an entity not in the class.

For $A2$. $K =$ all real numbers. $\otimes = <$. $a \oplus b = b$.

For $A3$. $K =$ all real numbers. $\otimes = <$. $a \oplus b = 2(a + b)$ when a and b are both positive or both negative; otherwise $a \oplus b = a + b$.

For $A4$. $K =$ all real numbers. $\otimes = <$. $a \oplus b = 0$ when $a \neq b$; and $a \oplus a = a$.

For $A5$. $K =$ all positive real numbers. $\otimes = <$. $\oplus = +$.

For $A6$. $K =$ all positive real numbers, with 0. $\otimes = <$. $\oplus = +$.

For $RA1$. $K =$ all real numbers. $(a \otimes b) = (a > b)$ when a and b are both positive, otherwise $\otimes = <$. $\oplus = +$.

For $RA2$. $K =$ all real numbers. $(a \otimes b) = (a > b)$ when a and b are both negative, otherwise $\otimes = <$. $\oplus = +$.

* This is the now familiar method of proving independence.

Thus the independence of the sixteen postulates of § 3 is established.

The independence of the ten postulates of § 1 is proved as follows:

For postulates I, II, $R1$–$R6$, use the systems just given, omitting the specifications in regard to $\oplus$.

For postulate $R7$, namely, $a, b \epsilon K. \ a \neq b. \supset . \ \exists K \frown x \ni (x \otimes a)$, use the system: $K =$ all real numbers between 0 and 1, with 0; $\otimes = <$.

For postulate $R8$, namely, $a, b \epsilon K. \ a \neq b. \supset . \ \exists K \frown y \ni (y \otimes a)$, use the system: $K =$ all real numbers between -1 and 0, with 0; $\otimes = <$.

To prove the independence of the eight postulates of § 2, use the systems above, omitting the specifications in regard to $\otimes$.

§ 5. Further analysis of the postulates for group-theory.*

The set of eight postulates given in § 2 can be used only for *abelian* groups. In the present section I give a set of twelve postulates for groups in general, with a thirteenth postulate which makes the group abelian; and moreover (except in the case of postulate $G9$, which can probably be further sub-divided), each postulate has been made as nearly a *simple statement* as seems possible.†

Since the theory of groups has many applications not connected with real algebra, I represent the fundamental operation of the group, in the present set of postulates, by the symbol $\circ$, or by simple juxtaposition, instead of by the symbol $+$ or $\oplus$. Thus, $a \circ b$ (read: "a with b"), or ab, may be interpreted at pleasure as the "sum" of a and b, or the "product" of a and b, or the "resultant" of a and b, etc. I also represent the "zero-element," 0 or z, of § 2, by the symbol i ("identity"), which has a less special connotation.‡

(The letter G is prefixed to the numbers of the postulates to suggest the word Group.)

Postulates I–II. The same as in § 2. Hence:

Theorem I. *There are at least two distinct elements in the class.*

Postulate $G1$. If a is an element of the class, then aa is an element of the class.

Postulate $G2$. If $a \neq b$, then ab is an element of the class.—Hence:

Theorem II. *If a and b are elements of the class, then ab is an element of the class.*

* Another set of postulates for groups will be presented in a subsequent paper.

† Most of the postulates embody, to be sure, a multitude of statements, corresponding to the multitude of elements in the group; but (except in case of $G9$) there seems to be no ground for distinguishing any of these statements from the rest; that is, there seems to be no ground for further subdivision of any of the postulates except $G9$. Cf. E. H. Moore, loc. cit.

‡ Compare footnote under postulate $A1$ in § 2.

POSTULATE $G3$. If there is any element in the class, there is an element i such that $ii = i$.

POSTULATE $G4$. If x and y are elements such that $xx = x$ and $yy = y$, then $x = y$. Hence:

Theorem III. *There is a uniquely determined element i such that $ii = i$.*

POSTULATE $G5$. If $ii = i$ and $aa \neq a$, then $ai = a$.

POSTULATE $G6$. If $ii = i$ and $aa \neq a$, then $ia = a$. Hence:

Theorem IV. *For every element a we have $ai = ia = a$.*

POSTULATE $G7$. If $ax = a$ and $aa \neq a$, then $xx = x$.

POSTULATE $G8$. If $xa = a$ and $aa \neq a$, then $xx = x$. Hence:

Theorem V. *If $ax = a$, or $xa = a$, then $x = i$.*

POSTULATE $G9$. If $aa \neq a$, $bb \neq b$, $cc \neq c$, and $ab \neq a$, $ab \neq b$, $bc \neq b$, $bc \neq c$; then

$$(ab)c = a(bc).$$

Hence:

Theorem VI. *For all elements $(ab)c = a(bc)$.* (Associative law.)

POSTULATE $G10$. If $ii = i$ and $aa \neq a$, then there is either an element a_r such that $aa_r = i$, or an element a_l such that $a_l a = i$.*

Now from $ax = i$ follows $xa = i$, and conversely. Thus, if $ax = i$, then $a(xa) = (ax)a = ia = a$, whence $xa = i$, by theorem V.

Further, if $ax = i$ and $ay = i$, then $x = y$. For, from $ax = i$ follows $yax = yi = y$, and from $ay = i$ follows $ya = i$ and $yax = ix = x$.

Hence, from postulate $G10$,

Theorem VII. *Every element a determines uniquely an element $\bar{a}$ such that $a\bar{a} = \bar{a}a = i$.*

Then if $ax = ay$, we have $\bar{a}ax = \bar{a}ay$, whence $x = y$; and similarly if $xa = ya$, we have $xa\bar{a} = ya\bar{a}$, whence $x = y$. Hence:

Theorem VIII. *A change in either a or b alone produces a change in ab.*

These twelve postulates, I–II, $G1$–$G10$, are thus sufficient to define a *group*. In order to make this group an *abelian* group, we must add another postulate, namely:

POSTULATE $G11$. If $aa \neq a$, $bb \neq b$, $ab \neq a$, $ab \neq b$, $ba \neq a$, and $ba \neq b$, then

$$ab = ba.$$

Hence:

Theorem IX. *For all elements, $ab = ba$.* (Commutative law.)

These thirteen postulates, I–II, $G1$–$G11$, are *independent* of each other, as is shown by the following systems, or interpretations of K and $\oplus$:

For I *and* II. The same systems as in § 4.

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* This postulate was suggested by E. H. MOORE's six-postulate definition of a group; Transactions, vol. 3 (1902), p. 489; vol. 5 (1904), p. 549.

*For G*1. $K=$ all real numbers. $a \oplus b = a + b$ when $a \neq b$, or $a = 0$, or $b = 0$, or $a + b = 0$; otherwise $a \oplus b$ not in the class.

*For G*2. $K=$ all real numbers. $a \oplus b = a + b$ when $a = b$, or $a = 0$, or $b = 0$, or $a + b = 0$; otherwise $a \oplus b$ not in the class.

*For G*3. $K=$ all positive real numbers. $\oplus = +$.

*For G*4. $K=$ all positive real numbers. When $a \neq b$, $a \oplus b =$ the greater of the two numbers a and b; when $a = b$, $a \oplus a = a$.

*For G*5. $K=$ all positive real numbers with 0. $a \oplus b = a + b$, except that $a \oplus 0 = 0$.

*For G*6. $K=$ all positive real numbers with 0. $a \oplus b = a + b$, except that $0 \oplus a = 0$.

*For G*7. $K=$ all real numbers. $a \oplus a = 0$, $0 \oplus a = a \oplus 0 = a$; otherwise $a \oplus b = a$.

*For G*8. $K=$ all real numbers. $a \oplus b = 0$, $0 \oplus a = a \oplus 0 = a$; otherwise $a \oplus b = b$.

*For G*9. $K=$ all real numbers. $a \oplus b = 2(a + b)$, when a and b are both positive or both negative; otherwise $a \oplus b = a + b$.

*For G*10. $K=$ all positive real numbers with 0. $\oplus = +$.

*For G*11. Any non-abelian group with respect to $\oplus$. For example: $K=$ the class of all complex numbers (α, β) where α is positive and β real, with $\oplus$ defined so that $(\alpha_1, \beta_1) \oplus (\alpha_2, \beta_2) = (\alpha_1 \alpha_2, \alpha_2 \beta_1 + \beta_2)$. The complex numbers (α, β) may be represented by the points of the $\alpha\beta$-plane which lie to the right of the β-axis, the element i being the point $(1, 0)$.

§ 6. Another set of postulates for real algebra.

The set of postulates for groups given in § 5 suggests at once, (if we replace* ab by $a + b$ and i by 0), another set of postulates for real algebra, one which does not involve the commutative law,— namely the set consisting of the following twenty postulates:

Postulates I–II. (As in § 1 or § 2.)—These postulates give the class at least two distinct elements.

Postulates *R*1–*R*6. (As in § 1.)—These postulates, with I and II, make the class a *continuous scale*, or a *one-dimensional continuum*, with respect to the relation $<$.

Postulates *G*1–*G*10. (As in § 5.)—These postulates, with I and II, make the class a *group* (not necessarily an abelian group) with respect to the operation $+$.

Postulate *RG*1. If there is a uniquely determined element 0 such that $0 + 0 = 0$; and if a, b, $a + b$, and $b + a$ are elements of the class such that $a + b \neq a$, $a + b \neq b$, $b + a \neq a$, and $b + a \neq b$; then: if a and b are > 0, then $a + b$ is $> a$ and $> b$.

* Compare footnote under postulate *A*1 in § 2.

POSTULATE $RG2$. If the hypothesis of $RG1$ holds true, then: if a and b are < 0, then $a + b$ is $< a$ and $< b$.

These last two postulates serve to connect the symbols $<$ and $+$. In view of Postulates I, II, and $G1$–$G10$ [see especially theorem V in § 5] they give at once theorem 10 of § 3, namely: *If a and b are both above* 0, *then* $a + b$ *is above either of them; and if a and b are both below* 0, *then* $a + b$ *is below either of them.*

It remains only to prove the commutative law. From theorem 10 we have, as in § 3, theorems 13 and 15, and also:

Theorem 13a. *If* $x > 0$, *then* $x + a > a$; *and if* $x < 0$, *then* $x + a < a$; and

Theorem 15a. *If* $x < y$, *then* $x + a < y + a$, *and conversely.*

From these theorems the principle of ARCHIMEDES can be deduced (as in theorem 23) without the aid of the commutative law, and hence that law itself can be proved by the method employed in my paper on the postulates of magnitude.*

Thus the equivalence of the sets of postulates in § 3 and § 6 is established.

The *independence* of the twenty postulates of this set (§ 6) is shown by the following systems:

For I–II, *and* $R1$–$R6$. The same systems as in § 4.

For $G1$–$G10$. The systems used in § 5, with $\circledcirc$ defined as $<$.

For $RG1$ *and* $RG2$. The systems used for $RA1$ and $RA2$ in § 4.

An unsolved problem.†

I conclude this section with the statement of the following problem: Suppose that the postulates $RG1$ and $RG2$ in § 6 are "weakened" so as to read "$\cdots > a$ OR $> b$" and "$\cdots < a$ OR $< b$," instead of "$\cdots < a$ AND $< b$" and "$\cdots < a$ AND $< b$"; is the commutative law then still deducible, or is postulate $G11$ an independent postulate?

In attempting to prove the latter alternative, two "proof-systems" suggest

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* Transactions, vol. 3 (1902), pp. 270–271. The method is due to O. HÖLDER.—The same method is used by BURALI-FORTI (*Teoria generale* $\cdots$, loc. cit.) in deducing the commutative law from a set of eight postulates for the class of positive real numbers with 0. In order, however, to make his proof of the principle of ARCHIMEDES convincing, certain modifications are necessary. In a letter of November 11, 1904, he proposes, therefore, the following corrected form of his definition 1·3, page 6, and his postulate VI, page 7:

$$\cdot 3.\ a\varepsilon u\,.\supset.\ \theta(u,f)a = u\frown x\ni[a\varepsilon\{xf(u\text{-Nul}(u,f))\smile(u\text{-Nul}(u,f))fx\}] \qquad \text{Df}$$

$$[\text{VI}\,|\,x,\ y\varepsilon u\,.\supset_{x,y}: x = y\,.\smile.\ x\varepsilon\theta(u,f)y\,.\smile.\ y\varepsilon\theta(u,f)x\,.\because$$

With these changes, his proofs of the principle of ARCHIMEDES and the commutative law present no further difficulty.

† This problem is closely connected with the unsolved problem No. 1 in my paper on magnitudes, Transactions, vol. 3 (1902), p. 279.

themselves. First, we might try the system used for $G11$ in § 5, with $\otimes$ defined as follows: $(\alpha_1, \beta_1) \otimes (\alpha_2, \beta_2)$ when $\alpha_1 < \alpha_2$, or when $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$. This system satisfies $RG1$ and $RG2$, but fails for $R6$ (on continuity). Again, we might try the same system as to K and $\oplus$, with $\otimes$ defined as follows: bring the points (α, β) of the half-plane into one-to-one correspondence with the points of a straight line as G. CANTOR has shown how to do;* and then interpret $\otimes$ as "below" (or "on the left of"). This system satisfies all the postulates $R1$–$R6$, but it is not easy to see how to make it satisfy *both* of the postulates $RG1$ and $RG2$.—The independence of postulate $G11$, in the revised set of postulates proposed in the problem, is therefore an open question.

§ 7. A CONDENSED LIST OF POSTULATES.

In the previous sections I have attempted to analyze the fundamental propositions of real algebra into their simplest (independent) component statements, and the number of postulates in each list is therefore large. In the present section, on the other hand, I give a condensed list of postulates, in which the number is much smaller. The list comprises the following ten postulates:

1. There are at least two distinct elements in the class.

2. If $a < b$ and $b < c$, then $a < c$.

3. $a < a$ is false.

4. If $a \neq b$, then there is either an element x such that $a < x$ and $x < b$, or an element y such that $a > y$ and $y > b$.

5. If Γ and Γ' are two non-empty sub-classes in K, such that every element of Γ is $<$ every element of Γ', then there is at least one element, M, which is $\geqq$ every element of Γ and $\leqq$ every element of Γ'.

6. If a and b are elements of the class, then $a + b$ is an element of the class.

7. $a + b = b + a$, whenever the sums involved are elements of the class.

8. $(a + b) + c = a + (b + c)$, whenever the sums involved are elements of the class.

9. If a, b, and $a + b$ are elements of the class, there is an element x such that $a = b + x$.

10. If $x < y$ then $a + x < a + y$, whenever the sums involved are elements of the class.

The sufficiency of these ten postulates is shown as follows:

From postulates 2 and 3 we have: $a < b$ and $a > b$ cannot both be true. From postulates 2 and 4 we have: if $a \neq b$, then either $a < b$ or $a > b$. Postulates 4 and 5 give density and continuity. Hence by § 1, postulates 1–5

* G. CANTOR, *Ein Beitrag zur Mannigfaltigkeitslehre*, CRELLE'S Journal für die reine und angewandte Mathematik, vol. 84 (1877), pp. 242–258; translated under the title *Une contribution à la théorie des ensembles*, Acta Mathematica, vol. 2 (1883), pp. 311–328.

make the class a *continuous scale*, or a *one-dimensional continuum*, with respect to $<$. Further, postulates 6–9 make the class an *abelian group* * with respect to $+$; and postulate 10 gives theorem 10 of § 3.

Hence the set of ten postulates in § 7 defines the same algebra as the set of sixteen postulates in § 3.

The *independence* of postulates 1–10 is shown as follows: For 1–5, and 7–10, use the systems given in § 4 for postulates II, $R4$, $R3$, $R5$, $R6$, and $A2$, $A3$, $A5$, $RA1$, respectively. For 6, use the system: $K =$ all real numbers; $\circledless = <$; $a \oplus b = a + b$ when a and b are integers; otherwise $a \oplus b$ not in the class.

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September, 1904.

APPENDIX. A SET OF POSTULATES FOR REAL ALGEBRA, IN WHICH MULTIPLICATION IS INCLUDED AS A FUNDAMENTAL CONCEPT.

In the preceding sections of this paper, multiplication has been thought of as a derived concept, the product of two elements with respect to an arbitrarily chosen (positive) unit-element being definable in terms of the fundamental concepts $<$ and $+$. In this appendix, on the other hand, multiplication is regarded as itself a fundamental concept, whose properties are determined by postulates.

We consider, namely, a class, K, a relation, $<$ (read: "below" or "before") and *two* operations, $+$ and $\times$ (read: "plus" and "times"), and inquire what conditions must be imposed upon the system $(K, <, +, \times)$ to make it equivalent to the algebra of a real variable with respect to $<$, $+$, and $\times$. These conditions are expressed in the following twenty postulates (the letters R, A, and M indicating the postulates which concern the Relation $<$, the operation of Addition, and the operation of Multiplication, respectively):

POSTULATES I, II, $R1$–$R6$, $A1$–$A6$, $RA1$–$RA2$. The same as in § 3.

POSTULATE $M1$. If a and b are elements of the class, then $a \times b$ (or ab) is an element of the class.

POSTULATE $M2$. If a, b, ab, and ba are elements of the class, then $ab = ba$. (Commutative law for multiplication.)

POSTULATE $AM1$. If $a, b, c, b + c, ab, ac, a(b + c)$, and $(ab) + (ac)$ are elements of the class, then $a(b + c) = (ab) + (ac)$. (Distributive law for multiplication with respect to addition.)

POSTULATE $RAM1$. If 0 is a uniquely determined element such that

* E. V. HUNTINGTON, Transactions, vol. 4 (1903), p. 27. But note that E. H. MOORE's query as to "definition W_1'," in Transactions, vol. 3 (1902), p. 489, is not yet completely answered.

$0+0=0$, and if a, b, and ab are elements of the class, and $a > 0$ and $b > 0$, then $ab > 0$.

From these postulates we have:

Theorem 31. $c \times 0 = 0 \times c = 0$ *for every element c.*

Theorem 32. $a \times (-b) = (-a) \times b = -(a \times b)$.

Theorem 33. *If a and b are both positive or both negative, ab is positive; if either is positive and the other negative, ab is negative.* (See definitions under theorem 12.) Hence:

Theorem 34. *If* $a \neq 0$ *and* $b \neq 0$, *then* $ab \neq 0$.

If now we select at pleasure a positive element E, and notice that every positive element a is the upper limit of the class of all the rational fractions of E which are $< a$ (see definitions under postulate $R6$ and theorem 27), we can prove the associative law for multiplication (first for positive elements, and then, by the aid of theorem 32, for all elements), and also the possibility of "division" when the divisor is different from zero; that is:

Theorem 35. $(ab)c = a(bc)$; and

Theorem 36. *For every two elements a and b, provided a is not zero, there is an element x such that* $ax = b$.

Thus we see that the elements of K *excluding zero* from an *abelian group* with respect to $\times$. [Hence:* 1°) there is a uniquely determined "unit-element," 1, not zero, such that $1 \times 1 = 1$; 2°) for every element a, $1 \times a = a \times 1 = a$; and 3°) if $a \neq 0$, then from $ax = ay$ follows $x = y$.] Therefore the whole system is a *field* † with respect to $+$ and $\times$.

Making use again of classes of rational fractions, we can now prove:

Theorem 37. *Any two systems,* $(K, <, +, \times)$ *and* $(K', <, +, \times)$, *which satisfy the twenty postulates of this appendix are* EQUIVALENT *with respect to* $<$, $+$ *and* $\times$; *that is, they can be brought into one-to-one correspondence in such a way that*

1°) *from* $a < b$ *follows* $a' < b'$, *and conversely;*

2°) *from* $a + b = c$ *follows* $a' + b' = c'$, *and conversely; and*

3°) *from* $ab = c$ *follows* $a'b' = c'$, *and conversely;—*

where a', b', c' *are the elements in* K' *which correspond to the elements* a, b, c *in* K. *Further, this correspondence can be established in only one way, since the "unit-element" is fixed in each system.*

The last clause of the theorem, namely that the correspondence can be established in only one way, marks the essential distinction between the postulates of the appendix (expressed in terms of $<$, $+$, and $\times$) and the postulates of

* Cf. Transactions, vol. 4 (1903), pp. 27–28.

† Cf. L. E. Dickson and E. V. Huntington, Transactions, vol. 4 (1903), pp. 13 and 31. Notice that postulates I, II, $A1$–$A6$, $M1$–$M2$, $AM1$ are not sufficient to define a field, without the aid of the postulates concerning $<$ (as witness the system of all integers with respect to $+$ and $\times$). A further discussion of the postulates for a field will be presented in a later paper.

§ 3 (expressed in terms of $<$ and $+$). For in § 3 the correspondence could be established in an infinite number of ways, according to the infinite number of ways in which the "unit-elements" might be selected. The postulates of the appendix, therefore, determine the algebra *uniquely* in a stronger sense than do the postulates of the preceding sections.

As examples of systems (K, $\otimes$, $\oplus$, $\odot$) which satisfy all the postulates, and to illustrate theorem 37, I mention the following "equivalent" systems (the symbols $<$, $+$, and $\times$, being now used in their ordinary arithmetical meanings):*

1) $K =$ all real numbers; $\otimes = <$; $\oplus = +$; $\odot = \times$. (Here $z = 0$, $u = 1$.)

2) $K =$ all real numbers > 0; $\otimes = <$; $a \oplus b = ab$; $a \odot b = \exp(\log a \cdot \log b)$. (Here $z = 1$, $u = e =$ base of the system of logarithms.)

3) $K =$ all real numbers > 1; $\otimes = <$; $a \oplus b = \exp(\log a \cdot \log b)$; $a \odot b = \exp\{\exp[\log(\log a) \cdot \log(\log b)]\}$. (Here $z = e$, $u = e^e$.)

4) $K =$ all real numbers; $\otimes = <$; $a \oplus b = (a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$; $a \odot b = ab$. (This example was suggested to me by Professor Bouton.)

The existence of any such system proves the *consistency* of the postulates.

It remains to show that the twenty postulates of the appendix are *independent*; that is, to exhibit twenty interpretations of K, $\otimes$, $\oplus$, and $\odot$, in which all the other postulates are satisfied, but not the one in question. These proof-systems are the following:

For I. $K =$ an "empty" class.

For II. $K =$ a class of a single element a. $a \otimes a$ false. $a \oplus a = a$. $a \odot a = a$.

For $R1$–$R6$, $A1$–$A6$, $RA1$–$RA2$. The systems used in § 4, with $\odot$ defined as ordinary multiplication, except that in the system for $R4$ the product is taken modulo 3.

For the remaining postulates, use the system of all real numbers, with $\otimes$ and $\oplus$ defined as the ordinary $<$ and $+$, and $\odot$ defined as follows:

For $M1$. $a \odot b = ab$ when a and b are both rational, otherwise $a \odot b$ not in the class.

For $M2$. $a \odot b = b$.

For $AM1$. $a \odot b = a + b$.

For $RAM1$. $a \odot b = -ab$.

Thus the independence of the twenty postulates is established.

We notice in conclusion that we may replace the first sixteen postulates of this list by the twenty postulates of § 6, or by the ten postulates of § 7, thus forming two new sets of twenty-four and fourteen postulates, respectively. The independence of the postulates of each of these sets is readily verified, by the aid of the proof-systems already employed.

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* See the opening paragraph in § 1 and § 2. For convenience in printing, exp x is written for e^x.